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Universal integrals of motion and universal invariants of quantum systems

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Abstract. Universal quantum integrals of motion are introduced, and their relation with the universal quantum invariants is established. The invariants concerned are certain combinations of the second- and higher-order moments (variances) of quantum-mechanical operators, which are preserved in time independently of the concrete form of the coefficients of the Schrödinger equation, provided the Hamiltonian is either a generic quadratic form of the coordinate and momenta operators, or a linear combination of generators of some finite-dimensional algebra (in particular, any semisimple Lie algebra). Using the phase space representation of quantum mechanics in terms of the Wigner function, the relations between the quantum invariants and the classical universal integral invariants by Poincaré and Cartan are elucidated. Examples of the 'universal invariant solutions' of the Schrödinger equation, i.e. self-consistent eigenstates of the universal integrals of motion, are given. Applications to the physics of optical and particle beams are discussed.

1. Introduction

The role of *integrals of motion* in physics can hardly be overestimated. They help to analyse and classify the behaviour of various classical and quantum systems. In particular, knowledge of integrals of motion simplifies significantly the process of solving dynamical equations governing the system evolution. In the quantum case, this was shown distinctly by Lewis and Riesenfeld [1], whose method was generalized and applied to different problems in numerous publications. A quantum integral of motion is defined usually as an operator whose average value $\langle\psi(t)|\hat{I}(t)|\psi(t)\rangle$ does not depend on time for any state $|\psi(t)\rangle$ obeying the Schrödinger equation. $\hat{I}(t)$ satisfies the equation $\mathrm{i}\hbar\partial\hat{I}/\partial t = \left[\hat{H}, \hat{I}\right]$, therefore its explicit form depends on the form of the Hamilton operator $\hat{H}$ (which is supposed to be Hermitian).

For example, in the simplest cases of a quantum harmonic oscillator with a time-dependent frequency $\omega(t)$ or a charged particle moving in a time-dependent homogeneous magnetic field, one has *linear* integrals of motion of the form [2–4]

$$\hat{A}(t) = \varepsilon(t)\hat{p} - \dot{\varepsilon}(t)\hat{x} \tag{1.1}$$

where $\varepsilon(t)$ is a solution to the classical equation $\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0$ (here $\hbar = m = 1$). Depending on the choice of the concrete solution $\varepsilon(t)$, eigenstates of the operator $\hat{A}(t)$ may be either generalized coherent states [2–5], or squeezed correlated states [5, 6], or propagators in various representations [7, 8]. *Quadratic* integrals of motion such as $\hat{A}^2(t)$ have been used

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in introducing *even and odd coherent states* [9], which are so popular nowadays due to their interpretation as examples of 'Schrödinger cat states'. Using quadratic integrals of motion such as

$$\hat{A}^{\dagger}(t)\hat{A}(t) = |\varepsilon|^2\hat{p}^2 + |\dot{\varepsilon}|^2\hat{x}^2 - \mathrm{Re}(\dot{\varepsilon}\varepsilon^*)(\hat{p}\hat{x} + \hat{x}\hat{p}) \tag{1.2}$$

(known as the 'Courant–Snyder invariant' [10] in particle beam physics and as the 'Lewis invariant' in quantum mechanics [11]), one can find time-dependent solutions to the Schrödinger equation, which are generalizations of the Fock states [1–4, 12]. Such integrals of motion are very useful for calculating the Berry phase [13–17]. They can also be generalized to the relativistic case [18]. For a detailed review on the method of quantum integrals of motion see, for example, [19].

All the operators discussed above (and their eigenstates) depend explicitly on functions such as $\varepsilon(t)$, i.e. after all, on the concrete form of the Hamiltonian (through the function $\omega(t)$, for instance, in the examples considered). It is known, however, that in classical mechanics there exist invariants of another type, which preserve their values in time simply due to the Hamiltonian structure of the equations of motion, *independently of the concrete form of the Hamilton function*. The most famous example is the preservation of the phase volume (the Liouville theorem), while the general construction is known under the name of *the universal integral invariants by Poincaré–Cartan* [20, 21]. These invariants are equal to the sums of the oriented hypervolumes (areas) of the projections of the chosen region in the phase space to all possible different subspaces (planes) generated by equal numbers of coordinate and momenta axes, such as $x_i p_i$, $x_i p_i x_j p_j$, etc. In quantum mechanics, there also exists a quantity which does not depend on time for any (Hermitian) operator: it is the integral $\int \psi^*(x)\psi(x)\,\mathrm{d}x$. However, this conservation law seems trivial.

The *quantum universal invariants*, which are conserved in time independently of the concrete form of coefficients of certain specific families of Hamiltonian operators, were discovered at the beginning of the 1980s [22]. In particular, such invariants exist for quantum systems described by Hamiltonians which are arbitrary inhomogeneous quadratic forms with respect to the canonical coordinates and momenta. These systems are distinguished, since their evolution is described in terms of *linear symplectic transformations*. Similar transformations also describe the evolution of paraxial light (laser) beams passing through linear optical systems, and the related universal invariants were studied in [23]. Although some results of papers [22, 23] were reproduced and generalized in reviews [19, 24], they still seem to be unknown to a wider audience. Therefore, the special cases of the invariants found in [22, 23] were rediscovered several times, although not in the quantum case, but in the theory of classical particle beams (with applications to accelerators) [25–27] and optical beams [28–37].

In this study we introduce the concept of *quantum universal integrals of motion*, as operators whose average values are universal invariants. The explicit forms of such operators include the variances of the quantum system, but they do not depend on the concrete coefficients of the Hamiltonian. Also, we introduce the self-consistent eigenstates of the universal integrals of motion, which can be called 'universal invariant solutions'. Besides, we provide new compact expressions for the universal invariants, including the invariants dependent on the higher-order statistical moments.

The plan of the paper is as follows. In section 2 we consider, following [22], the universal invariants of quantum systems, whose Hamiltonians are arbitrary quadratic forms with respect to the operators of the generalized coordinates and momenta. The relations between quantum universal invariants and the classical universal integral invariants by Poincaré and Cartan are elucidated, using the description of quantum systems in terms of the Wigner function, in section 3. The universal integrals of motion and 'universal invariant solutions' are studied in section 4, where the universal invariants containing higher-order moments are also considered. Section 5 is devoted to the general case where the Hamiltonians are linear with respect to generators of some algebra. Possible generalizations and applications in other areas, such as the theory of propagation of paraxial optical and particle beams, as well as some unsolved problems, are pointed out in section 6.

2. Universal invariants for quadratic Hamiltonians

2.1. General considerations

Consider a quantum system described in terms of a set of N operators, $\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_N$, obeying the bosonic commutation relations

$$\hat{Q}_\alpha \hat{Q}_\beta - \hat{Q}_\beta \hat{Q}_\alpha = \Sigma_{\alpha\beta} = -\Sigma_{\beta\alpha} \tag{2.1}$$

where $\Sigma_{\alpha\beta}$ is complex numbers forming the $N \times N$ antisymmetric matrix Σ. Let us assume that the dynamics of the system is governed by some *quadratic* Hamiltonian,

$$\hat{H} = \frac{1}{2} \sum_{j,k=1}^{N} B_{jk}(t) \hat{Q}_j \hat{Q}_k + \sum_{j=1}^{N} C_j(t) \hat{Q}_j \equiv \frac{1}{2} \hat{\boldsymbol{Q}} B(t) \hat{\boldsymbol{Q}} + \boldsymbol{C}(t) \hat{\boldsymbol{Q}} \tag{2.2}$$

where $B = \|B_{jk}\|$ is a symmetric $N \times N$ matrix with c-number elements, $\hat{\boldsymbol{Q}}$ is the N-dimensional vector, consisting of operators $\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_N$, and $\boldsymbol{C}$ is an N-dimensional c-number vector. In such a case, solutions to the Heisenberg equations of motion for the operators $\hat{Q}_\alpha(t)$ are expressed linearly through the initial operators $\hat{Q}_\alpha(0)$

$$\hat{Q}_\alpha(t) = \sum_\beta \Lambda_{\alpha\beta}(t) \hat{Q}_\alpha(0) + \chi_\alpha(t) \tag{2.3}$$

where $\Lambda_{\alpha\beta}(t)$ and $\chi_\alpha(t)$ are some c-number functions of time. An immediate consequence of equation (2.3) is the relation

$$Q_{\alpha\beta}(t) = \sum_{\mu,\nu} \Lambda_{\alpha\mu}(t) Q_{\mu\nu}(0) \Lambda_{\beta\nu}(t) \tag{2.4}$$

where

$$Q_{\alpha\beta} = \frac{1}{2} \langle \hat{Q}_\alpha \hat{Q}_\beta + \hat{Q}_\beta \hat{Q}_\alpha \rangle - \langle \hat{Q}_\alpha \rangle \langle \hat{Q}_\beta \rangle \tag{2.5}$$

is the second-order central moment (variance). We shall also use the equivalent notation for the variances: $Q_{\alpha\beta} \equiv \overline{Q_\alpha Q_\beta}$ and $\overline{Q_\alpha Q_\alpha} \equiv \overline{Q_\alpha^2}$ (without the summation over indices). The specific feature of quadratic systems is the independence of the dynamics of the variances $Q_{\alpha\beta}$ from the dynamics of the first-order mean values $\langle \hat{Q}_\alpha \rangle$ (i.e. from the functions $C_\alpha(t)$ and $\chi_\alpha(t)$).

Introducing the $N \times N$ matrix $\Lambda(t) = \|\Lambda_{\alpha\beta}(t)\|$ one can easily verify that it satisfies the linear equation

$$\dot{\Lambda} = -\frac{\mathrm{i}}{\hbar} \Sigma B \Lambda \tag{2.6}$$

which results in the equation $(\mathrm{d}/\mathrm{d}t)(\Lambda \Sigma \tilde{\Lambda}) = 0$, where $\tilde{\Lambda}$ denotes the transposed matrix. Taking into account the initial condition $\Lambda(0) = E_N$ (where E_N is the $N \times N$ unity matrix) we arrive at the fundamental identity

$$\Lambda(t) \Sigma \tilde{\Lambda}(t) \equiv \Sigma \tag{2.7}$$

which means that Λ is *symplectic* matrix. Another fundamental identity

$$\det \Lambda(t) \equiv 1 \tag{2.8}$$

is a consequence of the Liouville formula for the matrix determinant [38]

$$\det \Lambda(t) = \exp\left[\int_0^t \mathrm{Tr}\left(\dot{\Lambda}(\tau)\Lambda^{-1}(\tau)\right)\mathrm{d}\tau\right] \tag{2.9}$$

and the traceless property of the matrix ΣB: $\mathrm{Tr}(\Sigma B) = \mathrm{Tr}(\widetilde{\Sigma B}) = \mathrm{Tr}(\tilde{B}\tilde{\Sigma}) = -\mathrm{Tr}(\Sigma B) = 0$. (If the matrix Σ is non-degenerate, then (2.8) is an obvious consequence of (2.7) and the continuity of the matrix $\Lambda(t)$ in time. However, there exist interesting examples [22, 24] of the systems with the degenerate matrix Σ, for example, if the number N is *odd*.)

Comparing the matrix form of formula (2.4)

$$Q(t) \equiv \left\| Q_{\alpha\beta}(t) \right\| = \Lambda(t)Q(0)\tilde{\Lambda}(t) \tag{2.10}$$

with (2.7) it is easy to verify the identity

$$\mathcal{D}(\gamma; t) = \det\left[Q(t) - \gamma\Sigma\right] = \sum_{m=0}^{N} \mathcal{D}_m \gamma^m = \mathcal{D}(\gamma; 0) \tag{2.11}$$

where γ is an arbitrary auxiliary parameter. Consequently, the coefficients $\mathcal{D}_m$ do not depend on time. Moreover, they do not depend on the functions $B_{\alpha\beta}(t)$ and $C_\alpha(t)$ which determine the concrete form of the Hamiltonian. These coefficients depend only on the initial state of the system and the elements of the commutator matrix Σ. For these reasons we have called the conserved quantities $\mathcal{D}_m$ *quantum universal invariants* [22]. To find the number of independent invariants we note that the function $\mathcal{D}(\gamma)$ is *even*: $\det(Q - \gamma\Sigma) = \det\left(\tilde{Q} - \gamma\tilde{\Sigma}\right) = \det(Q + \gamma\Sigma)$. Therefore, the number of different independent invariants equals $[(N+1)/2]$ (since the coefficient $\mathcal{D}_N$ does not depend on $Q_{\alpha\beta}$ at all). If the Hamiltonian does not contain linear terms with respect to operators $\hat{Q}_\alpha$, then, besides the invariants $\mathcal{D}_m$, there also exists the invariants $\mathcal{D}'_m$, which can be obtained from $\mathcal{D}_m$ by means of the formal substitution $\langle\hat{Q}_\alpha\rangle = 0$ in the definition of the quantity $Q_{\alpha\beta}$.

2.2. Examples

For the set of usual momenta and coordinate operators, $\hat{\boldsymbol{Q}} = \left(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\right)$, the commutator matrix equals $\Sigma = -\mathrm{i}\hbar\Sigma_*$, where

$$\Sigma_* = \left\| \begin{array}{cc} 0 & E_n \\ -E_n & 0 \end{array} \right\|. \tag{2.12}$$

We shall use the notation $\mathcal{D}_j^{(n)}$, $j = 0, 1, \ldots, n-1$, in order to emphasize the dependence of the universal invariants on the number of degrees of freedom n. The simplest invariant is $\mathcal{D}_0^{(n)} = \det Q(t)$. In particular, in the case of one degree of freedom we have the universal invariant

$$\Delta \equiv \mathcal{D}_0^{(1)} = \overline{pp}\cdot\overline{xx} - (\overline{px})^2\,. \tag{2.13}$$

In the special case of the oscillator with a time-dependent frequency, this invariant was discovered in [39]. For $n \geqslant 2$ a relatively simple expression can be written for the invariant $\mathcal{D}_{2n-2}^{(n)}$, which is a quadratic form with respect to the variances $Q_{\alpha\beta}$:

$$\mathcal{D}_{2n-2}^{(n)} = \sum_{i,j=1}^{n} \left(\overline{p_i p_j}\cdot\overline{x_i x_j} - \overline{p_i x_j}\cdot\overline{x_i p_j}\right). \tag{2.14}$$

This invariant was also found in [40] (for $n = 1$ and 3). In the case of two degrees of freedom we have (writing $x_1 = x$, $x_2 = y$, $p_1 = p_x$, $p_2 = p_y$)

$$D_2^{(2)} = \overline{p_x^2}\,\overline{x^2} + \overline{p_y^2}\,\overline{y^2} + 2\overline{xy}\,\overline{p_x p_y} - 2\,\overline{xp_y}\,\overline{yp_x} - \left(\overline{yp_y}\right)^2 - (\overline{xp_x})^2 \tag{2.15}$$

$$\begin{aligned} D_0^{(2)} = {} & \left[\overline{x^2}\,\overline{y^2} - (\overline{xy})^2\right]\left[\overline{p_x^2}\,\overline{p_y^2} - \left(\overline{p_x p_y}\right)^2\right] + (\overline{xp_x})^2\,\left(\overline{yp_y}\right)^2 + (\overline{yp_x})^2\,\left(\overline{xp_y}\right)^2 \\ & -\overline{y^2}\,\overline{p_x^2}\left(\overline{xp_y}\right)^2 - \overline{x^2}\,\overline{p_y^2}\,(\overline{yp_x})^2 - \overline{y^2}\,\overline{p_y^2}\,(\overline{xp_x})^2 - \overline{x^2}\,\overline{p_x^2}\left(\overline{yp_y}\right)^2 \\ & -2\overline{xp_x}\,\overline{yp_y}\,\overline{xp_y}\,\overline{yp_x} - 2\overline{xy}\,\overline{p_x p_y}\left(\overline{xp_x}\,\overline{yp_y} + \overline{xp_y}\,\overline{yp_x}\right) \\ & +2\,\overline{xy}\left[\overline{p_x^2}\,\overline{yp_y}\,\overline{xp_y} + \overline{p_y^2}\,\overline{xp_x}\,\overline{yp_x}\right] + 2\,\overline{p_x p_y}\left[\overline{y^2}\,\overline{xp_y}\,\overline{xp_x} + \overline{x^2}\,\overline{yp_x}\,\overline{yp_y}\right]. \end{aligned} \tag{2.16}$$

Other invariants for $n > 2$ have too many terms to write here.

For the bosonic annihilation/creation operators, $\hat{\boldsymbol{Q}} = \left(\hat{a}_1, \ldots, \hat{a}_n, \hat{a}_1^\dagger, \ldots, \hat{a}_n^\dagger\right)$, $\left[\hat{a}_i, \hat{a}_k\right] = 0$, $\left[\hat{a}_i, \hat{a}_k^\dagger\right] = \delta_{ik}$, the commutator matrix is given by equation (2.12). In the one-dimensional case the analogue of invariant (2.13) reads [22]

$$\tilde{\Delta} = \sigma_N\,(\sigma_N + 1) - |\sigma_a|^2 = \text{constant} \tag{2.17}$$

where $\sigma_a = \langle\hat{a}^2\rangle - \langle\hat{a}\rangle^2$ and $\sigma_N = \langle\hat{a}^\dagger\hat{a}\rangle - \left|\langle\hat{a}\rangle\right|^2$. The analogue of the invariant (2.14) is

$$\tilde{\mathcal{D}}_{2n-2}^{(n)} = \sum_{i,j=1}^{n}\left(\overline{a_i a_j^\dagger}\cdot\overline{a_i^\dagger a_j} - \left|\overline{a_i a_j}\right|^2\right). \tag{2.18}$$

Examples of other sets of operators, with the commutator matrix different from (2.12) (e.g. kinetic momenta operators in the presence of a homogeneous magnetic field, when the number N may be odd), were given in [22, 24].

3. Geometric interpretation and the relations with the Poincaré–Cartan invariants

Since quantum systems with quadratic Hamiltonians are, in a certain sense, the closest to the classical ones [19, 41], it is natural to suppose that the quantum universal invariants are connected somehow with the classical Poincaré–Cartan integral invariants. Such relations do exist, and in the most distinct form they manifest themselves, if the quantum system is described in terms of the *Wigner function* [42, 43]

$$W(\boldsymbol{x}, \boldsymbol{p}) = \int \mathrm{d}\boldsymbol{v}\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{v}/\hbar}\langle\, \boldsymbol{x} - \boldsymbol{v}/2|\,\hat{\rho}\,|\boldsymbol{x} + \boldsymbol{v}/2\,\rangle \tag{3.1}$$

($\langle\boldsymbol{x}|\hat{\rho}|\boldsymbol{x}'\rangle$ is the matrix element of the statistical operator $\hat{\rho}$ in the coordinate representation). Consider the *Gaussian* Wigner function [24, 44, 45]

$$W(\boldsymbol{p}, \boldsymbol{x}) = \hbar^N (\det Q)^{-1/2} \exp\left[-\tfrac{1}{2}\left(\boldsymbol{q} - \langle\boldsymbol{q}\rangle\right) Q^{-1}(\boldsymbol{q} - \langle\boldsymbol{q}\rangle)\right] \tag{3.2}$$

where the vector $\boldsymbol{q}$ is defined as $\boldsymbol{q} = (\boldsymbol{p}, \boldsymbol{x})$, and Q is the corresponding symmetric variance matrix. In particular, in the one-dimensional case we have [6, 46]

$$W(x, p, t) = \Delta^{-1/2}\hbar \exp\left(-\frac{1}{2\Delta}\left[\sigma_{pp}(t)\tilde{x}^2 + \sigma_{xx}(t)\tilde{p}^2 - 2\sigma_{px}(t)\tilde{x}\tilde{p}\right]\right) \tag{3.3}$$

with Δ given by equation (2.13), $\sigma_{ab} \equiv \overline{ab}$, $a, b = x, p$, and

$$\tilde{x} = x - \langle \hat{x}(t) \rangle \qquad \tilde{p} = p - \langle \hat{p}(t) \rangle . \tag{3.4}$$

For quadratic Hamiltonians, any initial Gaussian state remains Gaussian for any $t > 0$; although the average values and variances of coordinates and momenta change with the course of time, the value of Δ is not changed. The lines of the fixed values of the quasiprobability $W =$ constant are the ellipses. Consider the ellipse corresponding to the value of the argument of the exponential in (3.3) equal to -1. The semiaxes of this ellipse are given by (in the system of units where x and p have the same dimensions)

$$a_{\pm} = \sqrt{T + \sqrt{\Delta}} \pm \sqrt{T - \sqrt{\Delta}} \qquad T = \tfrac{1}{2}\left(\sigma_{pp} + \sigma_{xx}\right).$$

The area of the ellipse equals $\pi a_{+} a_{-} = 2\pi\sqrt{\Delta}$. Consequently, equation (2.13) means nothing but the conservation of the phase volume (the phase area in the one-dimensional case) contained inside the surface of the constant Wigner quasiprobability, and in this sense the conservation of Δ it can be interpreted as the quantum analogue of the classical Liouville theorem. In the multidimensional case we have the invariant $\mathcal{D}_0^{(n)} = \det Q$, which is proportional to the square of the volume confined inside the surface of constant quasiprobability.

The conservation of the area of the ellipse in the phase space in the special case of the Gaussian wavepackets of free particles was mentioned in [47] (without any relation to the universal invariants). The fact that the ellipse of the constant quasiprobability, related to the Gaussian state of the oscillator (with constant frequency), rotates in the phase space without changing its shape, was noted as far back as in [48]. It is interesting to note that the universal invariants by Poincaré were used by Robertson [49] to illustrate the geometrical meaning of the generalized uncertainty relations, but he did not touch on the dynamical aspect of the problem.

The relations between the other invariants $\mathcal{D}_{2j}^{(n)}$ and the Poincaré–Cartan invariants are more involved. To elucidate them, it is sufficient to consider the Gaussian Wigner function with the diagonal variance matrix Q. Let us introduce the notation $\Delta_i = \overline{p_i^2} \cdot \overline{x_i^2}$. The invariants $\mathcal{D}_{2j}^{(n)}$ are equal in the case involved to the sums of all possible products of $(n - j)$ different factors Δ_i. Let us designate the Poincaré–Cartan invariants (for the chosen region confined with a surface of a constant value of the Wigner function) as $\mathcal{P}_m^{(n)}$, $m = 1, 2, \ldots, n$, the number $2m$ being equal to the dimensionality of the subspace which the chosen region in the phase space is projected to. In the case discussed, all the projections are ellipsoids whose semiaxes are proportional to the square roots of the variances $\overline{p_i^2}$ and $\overline{x_i^2}$. Therefore, the quantities $\mathcal{P}_m^{(n)}$ are proportional to the sums of all possible products of m different factors $\sqrt{\Delta_i}$. Taking into account the structure of the functions $\mathcal{D}_{2j}^{(n)}$ and $\mathcal{P}_m^{(n)}$ one can find the relation [22]

$$\mathcal{D}_{2j}^{(n)} = \sum_{k=0}^{\min(j,n-j)} \alpha_k \mathcal{P}_{n-j+k}^{(n)} \mathcal{P}_{n-j-k}^{(n)} \tag{3.5}$$

where α_k are some constant coefficients, and $\mathcal{P}_0^{(n)} \equiv 1$.

It is known that the evolution of the Wigner function in the case of a quadratic Hamiltonian (2.2) is described by the first-order partial differential equation [19, 24]

$$\frac{\partial W}{\partial t} = \sum_{k=1}^{2n} \frac{\partial}{\partial q_k} \left[(\Sigma_* \boldsymbol{B} \boldsymbol{q} + \Sigma_* \boldsymbol{C})_k \, W \right] \tag{3.6}$$

where $\boldsymbol{q} = (\boldsymbol{p}, \boldsymbol{x})$, and the matrix Σ_* is given by (2.12). Since (3.6) is nothing but the Liouville equation of the classical mechanics for the probability distribution, the Liouville theorem holds, in spite of the fact that the *quasiprobability* distribution $W(x, p)$ is not necessarily positive. Therefore, the volume of a region in the phase space confined with the surface $W =$ constant remains constant in time for *any* (not only Gaussian) initial Wigner function. This is a simple consequence of the fact that the propagator of the first-order partial differential equation (3.6) is the delta function [50], $G(\boldsymbol{q}, \boldsymbol{q}'; t) = \delta\left[\boldsymbol{q} - \Lambda(t)\boldsymbol{q}' - \boldsymbol{\chi}(t)\right]$, so the Jacobian of the transformation $\boldsymbol{q}' \to \boldsymbol{q}$ is equal to $\det \Lambda = 1$. Therefore, if the initial Wigner function was $W_0(\boldsymbol{q})$, then $W(\boldsymbol{q}, t) = W_0\left(\Lambda^{-1}(t)\left[\boldsymbol{q} - \boldsymbol{\chi}(t)\right]\right)$.

Not only the phase volume, but all other classical Poincaré–Cartan invariants are conserved for an *arbitrary* quantum initial state, if one applies these invariants to any region in the phase space confined with the surface $W =$ constant. This remarkable property is the direct consequence of the 'classical' evolution (3.6) for quantum systems with quadratic Hamiltonians. Moreover, this example shows the distinguished role of the Wigner function, compared with other 'quasiprobabilities' [43]. The evolution of all other quasiprobabilities (Husimi's, Cahill–Glauber's, etc) is governed by equations of the Fokker–Planck type, which contain the second-order spatial derivatives, giving an effective 'diffusion' in the phase space. Therefore, the Liouville theorem does not work for non-Wigner quantum distributions, as well as for quantum systems with non-quadratic Hamiltonians, since in these cases the evolution equations are no longer the classical Liouville equation.

4. Universal integrals of motion and universal invariant solutions

4.1. Universal integrals of motion and 'trace' invariants

In the Heisenberg picture, the vector $\hat{\boldsymbol{R}} = \hat{\boldsymbol{Q}} - \langle\hat{\boldsymbol{Q}}\rangle$ evolves as $\hat{\boldsymbol{R}}(t) = \Lambda(t)\hat{\boldsymbol{R}}(0)$. Consequently, the operator $\hat{\boldsymbol{I}}(t) = \Lambda^{-1}(t)\hat{\boldsymbol{R}}$ is the *integral of motion in the Schrödinger picture*, i.e. its average value $\langle\psi(t)|\hat{\boldsymbol{I}}(t)|\psi(t)\rangle$ does not depend on time for any state $|\psi(t)\rangle$ satisfying the Schrödinger equation (or a density matrix $\hat{\rho}(t)$ in the case of mixed states). The vector operator $\hat{\boldsymbol{I}}(t)$ is *linear* with respect to the operators $\hat{R}_\alpha$. Obviously, any *quadratic* integral of motion can be represented as

$$\hat{J} = \hat{\boldsymbol{I}}A\hat{\boldsymbol{I}} = \hat{\boldsymbol{R}}\tilde{\Lambda}^{-1}(t)A\Lambda^{-1}(t)\hat{\boldsymbol{R}} = \hat{\boldsymbol{R}}\tilde{\Sigma}^{-1}\Lambda(t)\tilde{\Sigma}A\Sigma\tilde{\Lambda}(t)\Sigma^{-1}\hat{\boldsymbol{R}} \tag{4.1}$$

where A may be an arbitrary symmetric constant matrix. The last equality in (4.1) is due to the consequences of the symplecticity condition (2.7)

$$\Lambda^{-1} = \Sigma\tilde{\Lambda}\Sigma^{-1} \qquad \tilde{\Lambda}\Sigma^{-1}\Lambda = \Sigma^{-1}. \tag{4.2}$$

The explicit form of the operator $\hat{J}$ depends on the matrix $\Lambda(t)$, i.e. on the concrete form of the quadratic Hamiltonian. However, comparing the last expression in (4.1) with the variance matrix transformation law (2.10), one recognizes immediately that matrices $\Lambda(t)$ and $\tilde{\Lambda}(t)$ can be removed if one identifies the matrix $\tilde{\Sigma}A\Sigma$ with $Q(0)$. Moreover, taking into account the consequence of equation (2.10)

$$Q(t)\left[\Sigma^{-1}Q(t)\right]^m = \Lambda(t)Q(0)\left[\Sigma^{-1}Q(0)\right]^m\tilde{\Lambda}(t) \tag{4.3}$$

one may identify the matrix $\tilde{\Sigma}A\Sigma$ with $Q(0)\left[\Sigma^{-1}Q(0)\right]^{2m}$ (the exponent $2m$ must be an even number, because the matrix $Q\left[\Sigma^{-1}Q\right]^{2m+1}$ is *antisymmetrical*, for a symmetrical matrix Q

and an antisymmetrical matrix Σ). Thus we obtain the family of integrals of motion

$$\hat{K}_{2m}(t) = -\hat{\boldsymbol{R}}\left[\Sigma^{-1}Q(t)\right]^{1+2m}\Sigma^{-1}\hat{\boldsymbol{R}} \tag{4.4}$$

whose coefficients are expressed through some integrals of the wavefunction $\psi(\boldsymbol{x},t)$ (or the density matrix), but they *do not depend on the concrete coefficients of the quadratic Hamiltonian*. It seems reasonable to call operators like (4.4) *universal integrals of motion*. One can easily verify that all operators (4.4) *commute between themselves*: $\left[\hat{K}_{2m},\hat{K}_{2n}\right]\equiv 0$ for any integers n and m. Note that integer m may be not only positive, but negative, as well, because the variance matrix Q cannot be singular, due to the generalized uncertainty relations [19, 22]. Introducing the notation $\hat{F}_{2m} = -\hat{K}_{-2-2m}$, we obtain an equivalent set of commuting integrals of motion

$$\hat{F}_{2m}(t) = \hat{\boldsymbol{R}}\left[Q^{-1}(t)\Sigma\right]^{1+2m}\Sigma^{-1}\hat{\boldsymbol{R}} \qquad \hat{F}_0(t) = \hat{\boldsymbol{R}}Q^{-1}(t)\hat{\boldsymbol{R}}. \tag{4.5}$$

The mean value of $\hat{F}_0(t)$ is trivial: $\langle\hat{F}_0(t)\rangle \equiv 2n$, where $2n$ is the dimensionality of the vector $\boldsymbol{R}$. However, for the mean value of the operator $\hat{K}_0(t)$ we have

$$\langle\hat{K}_0(t)\rangle = -\sum_{\alpha\beta\mu\nu} Q_{\alpha\beta}\left(\Sigma^{-1}\right)_{\beta\nu} Q_{\nu\mu}\left(\Sigma^{-1}\right)_{\mu\alpha} = -\operatorname{Tr}\left(Q\Sigma^{-1}Q\Sigma^{-1}\right). \tag{4.6}$$

Evidently, this is the *universal invariant*, since its value is conserved in time for any quadratic Hamiltonian, being independent of the explicit form of the Hamiltonian (although it depends on the initial quantum state). One can check that the right-hand side of (4.6) is proportional to the universal invariant $\mathcal{D}^{(n)}_{2n-2}$ (2.14). Moreover, calculating mean values of operators $\hat{K}_{2m}(t)$ or $\hat{F}_{2m}(t)$, we arrive at the set of 'trace' universal invariants

$$\mathcal{L}_{2m} = \operatorname{Tr}\left(\left[Q\Sigma^{-1}\right]^{2m}\right) \sim \langle\hat{K}_{2m-2}(t)\rangle \sim \langle\hat{F}_{-2m}(t)\rangle \tag{4.7}$$

which have been found for the first time (using another approach) in [22] and later rediscovered in [25, 26] (for the classical particle beams) and in [32, 33] (for the optical beams). The invariants in the form of the eigenvalues of matrix $Q\Sigma^{-1}$ were considered in [26, 28]. The 'trace' invariants of odd order $\mathcal{L}_{2m+1}$ equal zero identically, for any symmetrical matrix Q and antisymmetrical matrix Σ. Each invariant $\mathcal{L}_{2m}$ is some function of the invariants $\mathcal{D}_{2n-2j}$, $1 \leqslant j \leqslant m$. The invariants $\mathcal{L}_{2m}$ with $m > n$ (where $2n \times 2n$ is the dimension of the matrices) can be expressed in terms of the invariants with indices $m \leqslant n$, due to the Hamilton–Cayley theorem [38]. The same is true for the universal integrals of motion. Therefore, the independent 'trace' invariants are given by formula (4.7) with indices $m \leqslant n$.

4.2. Universal invariant solutions

Let us consider the special case $\hat{\boldsymbol{Q}} = \left(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n\right)$. It is convenient to renormalize integrals of motion and invariants using the matrix Σ_* (2.12) instead of the commutator matrix Σ defined in (2.1). Then we have a family of universal integrals of motion

$$\hat{K}^{(*)}_{2m,n}(t) = (-1)^m\hat{\boldsymbol{R}}\tilde{\Sigma}_*^{-1}\left[Q(t)\Sigma_*^{-1}\right]^{1+2m}\hat{\boldsymbol{R}} \tag{4.8}$$

where the second suffix n shows the number of degrees of freedom, while the coefficient $(-1)^m$ is introduced to ensure positiveness of average values (see below). In the one-dimensional case ($n = 1$) we have the following explicit expression for $m = 0$:

$$\begin{aligned}\hat{K}^{(*)}_{0,1}(t) = {} & \sigma_{pp}(t)\left(\hat{x} - \langle\hat{x}(t)\rangle\right)^2 + \sigma_{xx}(t)\left(\hat{p} - \langle\hat{p}(t)\rangle\right)^2 \\ & -\sigma_{px}(t)\left[\left(\hat{x} - \langle\hat{x}(t)\rangle\right)\left(\hat{p} - \langle\hat{p}(t)\rangle\right) + \left(\hat{p} - \langle\hat{p}(t)\rangle\right)\left(\hat{x} - \langle\hat{x}(t)\rangle\right)\right].\end{aligned} \tag{4.9}$$

This integral of motion was found in [39] in the special case of an oscillator with time-dependent frequency. For multidimensional systems, the integral of motion (4.4) with $m = 0$ was constructed in [22] and later in [51] (see also [36] for optical beams).

Another simple example is $\hat{F}^{(*)}_{0,1}(t) = \hat{K}^{(*)}_{0,1}(t)/\Delta$, where Δ is the universal invariant (2.13). This is exactly the same expression, which stands in the argument of the Gaussian Wigner function (3.3) (if one removes the operator symbols). Such a coincidence is not accidental, because the Wigner function, being a solution of the first-order partial differential equation (3.6), must depend on the *integrals of motion*, and not on the coordinates and time separately. All other universal integrals of motion are also proportional to $\hat{K}^{(*)}_{0,1}$ in the one-mode case: $\hat{K}^{(*)}_{2m,1} = \Delta^m \hat{K}^{(*)}_{0,1}$. Evidently, $\langle \hat{K}^{(*)}_{0,1}\rangle = 2\Delta$.

It is well known (see, e.g., [19]) that any *positively definite* quadratic one-mode Hamiltonian

$$\hat{H} = \mu\hat{p}^2 + \nu\hat{x}^2 + \rho\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) \tag{4.10}$$

can be reduced by means of a canonical transformation to the harmonic oscillator Hamiltonian with the effective frequency $\omega = \sqrt{\mu\nu - \rho^2}$. Consequently, its spectrum has the form $\hbar\omega(1 + 2s)$, $s = 0, 1, 2, \ldots$, and eigenfunctions in the coordinate representation are given by

$$\varphi_s(x) = \left(2^s s!\right)^{-1/2}\left(\frac{\omega}{\mu\pi\hbar}\right)^{1/4}\exp\left(-\frac{\omega + \mathrm{i}\rho}{2\hbar\mu}x^2\right)H_s\left(x\sqrt{\frac{\omega}{\hbar\mu}}\right) \tag{4.11}$$

where $H_s(z)$ is the Hermite polynomial. The operator $\hat{K}^{(*)}_{0,1}$ (4.9) is positively definite due to the generalized (Schrödinger–Robertson) uncertainty relation [6, 19, 52, 53]

$$\Delta \geqslant \hbar^2/4. \tag{4.12}$$

Therefore, the spectrum of $\hat{K}^{(*)}_{0,1}$ has the form $\kappa^{(0,1)}_s = \hbar\sqrt{\Delta}\,(1 + 2s)$ (independently of the first-order mean values $\langle\hat{p}\rangle$ and $\langle\hat{x}\rangle$). In general, eigenstates $|\varphi_s\rangle$ of the operator $\hat{K}^{(*)}_{(0,1)}$ have nothing in common with the time-dependent solution of the Schrödinger equation $|\psi(t)\rangle$ which determines the variances $\sigma_{pp}(t)$, $\sigma_{xx}(t)$ and $\sigma_{px}(t)$ in the right-hand side of equation (4.9). However, there are exceptional cases when the states $|\psi(t)\rangle$ and $|\varphi_s\rangle$ coincide (up to some phase factor, which can depend on time, generally speaking). This happens if the eigenvalue $\kappa^{(0,1)}_s$ is equal to the mean value of the operator $\hat{K}^{(*)}_{0,1}$ in the state $|\psi(t)\rangle$, i.e. $\hbar\sqrt{\Delta}\,(1+2s) = 2\Delta$. In this way, we find a set of discrete distinguished values of the universal invariant Δ (cf [54])

$$\Delta_s = \left[\hbar\left(s + \tfrac{1}{2}\right)\right]^2 \qquad s = 0, 1, 2, \ldots. \tag{4.13}$$

It seems natural to call the related eigenstates $|\psi_s(t)\rangle$ 'universal invariant solutions' of the Schrödinger equation. The eigenfunction of the operator $\hat{K}^{(*)}_{0,1}$ in the coordinate representation can be easily derived from equation (4.11), if one replaces the coefficients μ, ν, ρ by the variances in accordance with equation (4.9):

$$\psi_s(x, t) = \left(2^s s!\right)^{-1/2}\left(\frac{s + \frac{1}{2}}{\pi\sigma_{xx}}\right)^{1/4} H_s\left(\tilde{x}\sqrt{\frac{s + \frac{1}{2}}{\sigma_{xx}}}\right)$$
$$\times\exp\left[-\left(s + \tfrac{1}{2} - \mathrm{i}\sigma_{xp}/\hbar\right)\frac{\tilde{x}^2}{2\sigma_{xx}} + \frac{\mathrm{i}}{\hbar}\langle\hat{p}\rangle\tilde{x} + \mathrm{i}\Phi(t)\right]. \tag{4.14}$$

This wavefunction has a *non-invariant* time-dependent phase factor $\exp[\mathrm{i}\Phi(t)]$ (which depends on the concrete coefficients of the quadratic Hamiltonian) in order to satisfy the Schrödinger equation.

A more symmetrical and completely invariant solution can be written in the *Wigner representation*. Using the Groenewold [55] formula for the Wigner function of the harmonic oscillator energy eigenstate, or its generalization for eigenstates of quadratic integrals of motion [24, 56, 57], one obtains

$$W_s(x, p, t) = 2(-1)^s \mathrm{e}^{-y_s} L_s(2y_s) \tag{4.15}$$

where $L_s(z)$ is the Laguerre polynomial and

$$y_s(x, p, t) = \left[\hbar^2\left(s + \tfrac{1}{2}\right)\right]^{-1} \left[\sigma_{pp}(t)\tilde{x}^2 + \sigma_{xx}(t)\tilde{p}^2 - 2\sigma_{px}(t)\tilde{x}\tilde{p}\right] \tag{4.16}$$

(the symbols $\tilde{x}$ and $\tilde{p}$ were defined in equation (3.4)). Since $y_s(x, p, t)$ is a classical integral of motion, it is obvious that the function (4.15) satisfies the Liouville equation (3.6). Note that the first factor in the right-hand side of (4.16) equals $\left(\hbar\sqrt{\Delta_s}\right)^{-1}$. For $s = 0$ the function (4.15) coincides with the special case of the function (3.3) corresponding to $\Delta = \Delta_0 = \hbar^2/4$. This is the only possible value of the invariant Δ, when the Gaussian exponential (3.3) describes a pure quantum state [24].

The physical meaning of the solutions (4.14) and (4.15) is as follows. If one constructs the initial wavefunction $\psi_s(x, 0)$ or the Wigner function $W_s(x, p, 0)$ in accordance with (4.14) or (4.15), taking any values $\sigma_{pp}(0)$, $\sigma_{xx}(0)$ and $\sigma_{px}(0)$, obeying the constraint $\sigma_{pp}(0)\sigma_{xx}(0) - \sigma_{px}^2(0) = \Delta_s$, and quite arbitrary values $\langle\hat{x}(0)\rangle$ and $\langle\hat{p}(0)\rangle$, then the form of these functions will not be changed in the process of evolution, being given by the same expressions (4.14) or (4.15), but with the values of variances and the first-order moments related to the current instant of time t.

In the n-mode case, the universal invariant solutions are common eigenstates of n independent commuting quadratic operators $\hat{K}_0, \hat{K}_2, \ldots, \hat{K}_{2(n-1)}$. Here we give one family of such solutions, generalizing the set (4.15). Let us try to find an eigenstate of the operator $\hat{\boldsymbol{R}}U\hat{\boldsymbol{R}}$ in the Wigner–Weyl representation in the form $W(\boldsymbol{r}) = \exp(-y/2)\, f(y)$, where $y = \boldsymbol{r}G\boldsymbol{r}$, U, G are symmetrical matrices, $\boldsymbol{r} = \boldsymbol{q} - \langle\hat{\boldsymbol{Q}}\rangle$, and $f(y)$ is some function to be found. In the Wigner–Weyl representation the operator $\hat{\boldsymbol{Q}}$ has the realization [24] $\hat{\boldsymbol{Q}} = \boldsymbol{q} - (\mathrm{i}\hbar/2)\Sigma_*(\partial/\partial\boldsymbol{q})$. Therefore,

$$\begin{aligned}\hat{\boldsymbol{R}}U\hat{\boldsymbol{R}}W(\boldsymbol{r}) = \exp(-y/2)\big\{&f(y)\big[\boldsymbol{r}U\boldsymbol{r} + (\hbar^2/4)\boldsymbol{r}G\Sigma_*U\Sigma_*G\boldsymbol{r} - (\hbar^2/4)\,\mathrm{Tr}\,(\Sigma_*U\Sigma_*G)\,\big] \\ &+\hbar^2\boldsymbol{r}G\Sigma_*U\Sigma_*G\boldsymbol{r}f''(y) + f'(y)\big[(\hbar^2/2)\,\mathrm{Tr}\,(\Sigma_*U\Sigma_*G) - \hbar^2\boldsymbol{r}G\Sigma_*U\Sigma_*G\boldsymbol{r}\big]\big\}\end{aligned} \tag{4.17}$$

where $f'(y)$ denotes the derivative of the function $f(y)$ with respect to its argument. Taking $U = \tilde{\Sigma}_*^{-1}Q\Sigma_*^{-1}$ and $G = \gamma Q^{-1}$ (where γ is a constant number to be determined) one has $\Sigma_*U\Sigma_*G = -\gamma E_{2n}$. Then equation (4.17) assumes the form

$$\begin{aligned}\hat{K}_{0,n}^{(*)}W(\boldsymbol{r}) = \exp(-y/2)\big\{&f(y)\big[\boldsymbol{r}\tilde{\Sigma}_*^{-1}Q\Sigma_*^{-1}\boldsymbol{r} - (\hbar^2\gamma/4)y + (n\hbar^2\gamma/2)\big] \\ &-\hbar^2\gamma\big[yf''(y) + (n-y)f'(y)\big]\big\}.\end{aligned} \tag{4.18}$$

Remembering the equation for the associated Laguerre polynomials $F(z) = L_s^{(\alpha)}(z)$

$$zF''(z) + (\alpha + 1 - z)F' + sF = 0 \tag{4.19}$$

one can exclude the terms with derivatives from the right-hand side of (4.18) taking $f(y) = L_s^{(n-1)}(y)$. Then equation (4.18) becomes

$$\hat{K}_{0,n}^{(*)} W(\boldsymbol{r}) = W(\boldsymbol{r})\big\{\hbar^2\gamma(s+n/2) + \boldsymbol{r}\big[\tilde{\Sigma}_*^{-1} Q \Sigma_*^{-1} - (\hbar^2\gamma^2/4)Q^{-1}\big]\boldsymbol{r}\big\}.$$

The coordinate-dependent term in the right-hand side disappears for any matrix Q satisfying the condition (E_{2n} is the $2n \times 2n$ unit matrix)

$$\tilde{\Sigma}_*^{-1} Q \Sigma_*^{-1} Q = \frac{\hbar^2}{4}\gamma^2 E_{2n}. \tag{4.20}$$

To find the coefficient γ we demand that the eigenvalue $\hbar^2\gamma(s+n/2)$ coincide with the mean value of the operator $\hat{K}_{0,n}^{(*)}$ in the state $W(\boldsymbol{r})$ involved. Due to (4.20),

$$\langle \hat{K}_{0,n}^{(*)} \rangle = \mathrm{Tr}\left(\tilde{\Sigma}_*^{-1} Q \Sigma_*^{-1} Q\right) = \tfrac{1}{2}\hbar^2\gamma^2 n.$$

Thus we arrive at the discrete set of coefficients

$$\gamma_{s,n} = 1 + 2s/n \tag{4.21}$$

which yields the set of universal invariant solutions in the Wigner representation in the form (we omit the normalization factor)

$$W_{s,n}(\boldsymbol{r}) = \exp\left[-\left(\frac{1}{2} + \frac{s}{n}\right)\boldsymbol{r}Q^{-1}\boldsymbol{r}\right] L_s^{(n-1)}\left((1+2s/n)\boldsymbol{r}Q^{-1}\boldsymbol{r}\right). \tag{4.22}$$

These solutions are eigenstates of the operator $\hat{K}_{0,n}^{(*)}$,

$$\hat{K}_{0,n}^{(*)} W_{s,n} = \frac{\hbar^2}{2n}(2s+n)^2 W_{s,n} \tag{4.23}$$

provided the matrix $Q(t)$ satisfies the condition (4.20) (which is invariant with respect to the transformation $Q_0 \rightarrow \Lambda(t)Q_0\tilde{\Lambda}(t)$ for any symplectic matrix Λ). In the one-mode case the function (4.22) coincides with (4.15); moreover, in this case the matrix $\tilde{\Sigma}_*^{-1} Q \Sigma_*^{-1} Q$ is always proportional to the 2×2 unit matrix. If condition (4.20) is not fulfilled, then function (4.22) still satisfies the Liouville evolution equation (3.6) (as well as any other function of the classical integral of motion $\boldsymbol{r}Q^{-1}\boldsymbol{r}$), but it is not an eigenstate of the quantum integral of motion $\hat{K}_{0,n}^{(*)}$.

Due to condition (4.20), function (4.22) is also an eigenfunction for all operators $\hat{K}_{2m,n}^{(*)}$ with the eigenvalues

$$\kappa_{s,n}^{(m)} = 2n\left[\frac{\hbar}{2n}(2s+n)\right]^{2+2m}. \tag{4.24}$$

It would be interesting to find other families of universal invariant solutions in the n-dimensional case (one may suppose that the most general solution must depend on n parameters, in accordance with the existence of n independent integrals of motion) and to obtain an explicit expression for the counterpart of the solution (4.22) in the coordinate representation. We leave these problems for future studies.

4.3. Invariants containing higher-order moments

It is almost evident (and can be easily proved for *Hamiltonian* systems, i.e. in the absence of irreversible processes [19, 58]) that any function of integrals of motion is another integral of motion. Thus, taking powers of operators $\hat{K}_{2m}(t)$ or $\hat{F}_{2m}(t)$, we arrive at *non-quadratic integrals of motion* $\left[\hat{K}_{2m}(t)\right]^j$ and $\left[\hat{F}_{2m}(t)\right]^j$, whose average values, also being *universal invariants*, are expressed in terms of the moments of the order $2j$ (and the second-order variances):

$$\mathcal{K}_{2m}^{(j)} = \left\langle \left(\hat{\boldsymbol{R}}\tilde{\Sigma}^{-1}Q(t)\left[\Sigma^{-1}Q(t)\right]^{2m}\Sigma^{-1}\hat{\boldsymbol{R}}\right)^j\right\rangle \tag{4.25}$$

$$\mathcal{F}_{2m}^{(j)} = \left\langle \left(\hat{\boldsymbol{R}}\left[Q^{-1}(t)\Sigma\right]^{1+2m}\Sigma^{-1}\hat{\boldsymbol{R}}\right)^j\right\rangle. \tag{4.26}$$

In particular, in the one-dimensional case,

$$\begin{aligned}\mathcal{F}_0^{(2)} = \Delta^{-2}\left[\sigma_{pppp}\sigma_{xx}^2 + \sigma_{xxxx}\sigma_{pp}^2 + 6\sigma_{ppxx}\sigma_{px}^2 - 4\sigma_{pppx}\sigma_{xx}\sigma_{xp} - 4\sigma_{xxxp}\sigma_{pp}\sigma_{xp}\right] \\ +\Delta^{-1}\left\langle(\delta\hat{p})^2(\delta\hat{x})^2 + (\delta\hat{x})^2(\delta\hat{p})^2\right\rangle\end{aligned} \tag{4.27}$$

where the symbol $\sigma_{ab\ldots c}$ denotes the average value of the sum of all possible different products of operators $\delta\hat{a}, \delta\hat{b}, \ldots, \delta\hat{c}$ (where $\delta\hat{a} \equiv \hat{a} - \langle\hat{a}\rangle$), divided by the number of terms in this sum. For example,

$$\sigma_{ppxx} = \tfrac{1}{6}\left\langle(\delta\hat{p})^2(\delta\hat{x})^2 + (\delta\hat{x})^2(\delta\hat{p})^2 + \delta\hat{p}\delta\hat{x}\delta\hat{p}\delta\hat{x} + \delta\hat{x}\delta\hat{p}\delta\hat{x}\delta\hat{p} + \delta\hat{p}(\delta\hat{x})^2\delta\hat{p} + \delta\hat{x}(\delta\hat{p})^2\delta\hat{x}\right\rangle.$$

Such mean values can be written in the simplest form through the Wigner function:

$$\sigma_{ab\ldots c} = \int \frac{\mathrm{d}p\,\mathrm{d}q}{2\pi\hbar} W(p,q)\delta a\delta b\ldots\delta c \qquad a, b = p \text{ or } x.$$

A special case of the invariant (4.27) was considered (for optical paraxial beams) in [36], where the inequality $\mathcal{F}_0^{(2)} > 4$ was established. Similar invariants containing the moments of arbitrary degrees were studied in the context of the classical beam propagation problem in [27], and for optical beams in [35]; the key point was the use of tensors constructed from the higher-order moments, generalizing the second-order moment matrix Q.

4.4. Remark on the average values

We have defined the moments in equation (2.5) having in mind the most usual situation, when the operators $\hat{Q}_\alpha$ are Hermitian and their product must be symmetrized in order to obtain real average values. However, formula (2.4) and all its consequences discussed above remain valid under the definition $Q_{\alpha\beta} = \langle\hat{Q}_\alpha\hat{Q}_\beta\rangle - \langle\hat{Q}_\alpha\rangle\langle\hat{Q}_\beta\rangle$. Moreover, besides the standard definition of the mean value as $\langle\hat{Q}\rangle = \langle\psi|\hat{Q}|\psi\rangle$, one may also use quantities such as the transition amplitude $\langle\varphi(t)|\hat{Q}|\psi(t)\rangle$, where $|\varphi(t)\rangle$ is an arbitrary 'reference' state.

Our constructions remain valid for the *mixed* quantum states, described in terms of the statistical operator $\hat{\rho}$ (or the mutual coherence function in the optical case). Also, besides the standard definition $\langle\hat{q}_\alpha\hat{q}_\beta\rangle = \mathrm{Tr}\left(\hat{\rho}\hat{q}_\alpha\hat{q}_\beta\right)$, one may use more sophisticated ones, for example, [24, 59] $\langle\langle\hat{q}_\alpha\hat{q}_\beta\rangle\rangle = \mu^{-1}\,\mathrm{Tr}\left(\hat{\rho}^2\hat{q}_\alpha\hat{q}_\beta\right)$ or [23, 24] $\langle\langle\langle q_\alpha q_\beta\rangle\rangle\rangle = \mu^{-1}\,\mathrm{Tr}\left(\hat{\rho}\hat{q}_\alpha\hat{\rho}\hat{q}_\beta\right)$, where $\mu \equiv \mathrm{Tr}\,\hat{\rho}^2$ is the quantum mechanical 'purity'.

5. Arbitrary Lie algebras

The existence of universal invariants as discussed above is the consequence of three factors: the linearity of the Heisenberg equations of motion, the symplectic identity (2.7) and the unimodular identity (2.8). Linear equations of motion also arise for the Hamiltonians of the form

$$\hat{H} = \sum_{\nu} f^{\nu}(t)\,\hat{z}_{\nu} \tag{5.1}$$

provided operators $\hat{z}_{\nu}$ are generators of an arbitrary Lie algebra:

$$\left[\hat{z}_{\alpha},\ \hat{z}_{\beta}\right] = \mathrm{i}\hbar \sum_{\nu=1}^{N} c_{\alpha\beta}^{\nu}\hat{z}_{\nu} \qquad c_{\alpha\beta}^{\nu} = -c_{\beta\alpha}^{\nu} \tag{5.2}$$

$$\hat{z}_{\alpha}(t) = \sum_{\beta=1}^{N} \Lambda_{\alpha\beta}(t)\,\hat{z}_{\beta}(0) \tag{5.3}$$

$$\dot{\Lambda}_{\alpha\beta} = \sum_{\nu,\delta=1}^{N} c_{\alpha\nu}^{\delta} f^{\nu}(t)\Lambda_{\delta\beta}. \tag{5.4}$$

Using the Liouville formula (2.9) and equation (5.4) one can verify that the necessary and sufficient condition of the unimodularity of matrix $\Lambda(t)$ for arbitrary coefficients $f^{\nu}(t)$ is the validity of the set of equations

$$\sum_{\alpha=1}^{N} c_{\alpha\beta}^{\alpha} = 0 \qquad \beta = 1, 2, \ldots, N. \tag{5.5}$$

Instead of (2.7) we now have the identity

$$\Lambda(t)g\tilde{\Lambda}(t) = g \qquad \Lambda = \left\|\Lambda_{\alpha\beta}\right\| \qquad g = \left\|g_{\alpha\beta}\right\| \tag{5.6}$$

where $g_{\alpha\beta}$ are the elements of the Killing–Cartan tensor,

$$g_{\alpha\beta} = g_{\beta\alpha} = \sum_{\delta\rho} c_{\alpha\delta}^{\rho} c_{\beta\rho}^{\delta}. \tag{5.7}$$

To prove (5.6) we introduce the matrix $X = \Lambda g\tilde{\Lambda} - g$. Due to (5.4) it satisfies the equation

$$\dot{X} = AX + X\tilde{A} + Ag + g\tilde{A} \tag{5.8}$$

where the elements of matrix $A = \|A_{\alpha\beta}\|$ are given by $A_{\alpha\beta} = \sum_{\nu} c_{\alpha\nu}^{\beta} f^{\nu}$. However, $Ag + g\tilde{A} \equiv 0$, since the coefficients $c_{\alpha\nu\beta} = \sum_{\delta}\left(c_{\alpha\nu}^{\delta} g_{\delta\beta}\right)$ are antisymmetrical with respect to all three indices [60]. Consequently, the unique solution to the *homogeneous* equation (5.8) with zero initial condition is $X \equiv 0$. Note that the matrix $\Lambda(t)$ is unimodular for any semisimple algebra. Such algebras possess non-degenerate Killing–Cartan tensors [60], so the identity $\det\Lambda(t) \equiv 1$ is the immediate consequence of the identity (5.6).

Let us introduce the notation

$$z_{\alpha} = \left\langle\hat{z}_{\alpha}\right\rangle \qquad z_{\alpha\beta} = \tfrac{1}{2}\left\langle\hat{z}_{\alpha}\hat{z}_{\beta} + \hat{z}_{\beta}\hat{z}_{\alpha}\right\rangle - z_{\alpha}z_{\beta} \equiv z'_{\alpha\beta} - z''_{\alpha\beta}.$$

Introducing the matrix $Z = \left\|z_{\alpha\beta}\right\|$, one can easily verify that universal invariants are given by the coefficients of the expansion

$$\mathcal{G}(\gamma; t) = \det\left[Z(t) - \gamma g\right] = \sum_{m=0}^{N} \gamma^{m}\mathcal{G}_{m} = \mathcal{G}(\gamma; 0). \tag{5.9}$$

Using the quantities $z'_{\alpha\beta}$ or $z''_{\alpha\beta}$ instead of $z_{\alpha\beta}$ in (5.9), one obtains similar expressions $\mathcal{G}'_{m}$ and $\mathcal{G}''_{m}$, which are also universal invariants. Since both matrices, Z and g, are symmetrical, all coefficients in the expansion (5.9) are different from zero, so the number of independent invariants equals the number of algebra generators N. The equivalent set of 'trace' invariants can be written as (provided $\det g \neq 0$)

$$\mathcal{L}_m = \mathrm{Tr}\left(\left[Z(t)g^{-1}\right]^m\right) \qquad g^{-1} = \|g^{\alpha\beta}\|. \tag{5.10}$$

These invariants are average values of the universal integrals of motion

$$\hat{J}_n = \left(\hat{z} - \langle\hat{z}\rangle\right)\left(g^{-1}Z\right)^n g^{-1}\left(\hat{z} - \langle\hat{z}\rangle\right) \qquad \mathcal{L}_m = \langle\hat{J}_{m-1}\rangle \tag{5.11}$$

where $\hat{z} \equiv (\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_N)$. The exponent n may be an arbitrary integer (positive and negative, even and odd). However, only N integrals of motion are independent (for example, those given by equation (5.11) with $n = 1, 2, \ldots, N$). The simplest invariant coincides (up to a constant factor) with the mean value of the Casimir operator

$$\mathcal{G}'_{N-1} \sim \mathcal{L}'_1 = \mathrm{Tr}\left(Z'g^{-1}\right) = \left\langle\hat{z}_\alpha\hat{z}_\beta g^{\alpha\beta}\right\rangle. \tag{5.12}$$

Another simple invariant is $\mathcal{G}_0 \sim \det[Z(t)]$. It does not depend on the structure constants of the algebra.

5.1. Example

Consider the set of operators

$$\begin{aligned} &\hat{R}_1 = \hat{p}^2 && \left[\hat{R}_1,\ \hat{R}_2\right] = -4\mathrm{i}\hbar\hat{R}_3 \\ &\hat{R}_2 = \hat{x}^2 && \left[\hat{R}_2,\ \hat{R}_3\right] = 2\mathrm{i}\hbar\hat{R}_2 \\ &\hat{R}_3 = \tfrac{1}{2}\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right) && \left[\hat{R}_3,\ \hat{R}_1\right] = 2\mathrm{i}\hbar\hat{R}_1 \end{aligned} \tag{5.13}$$

which form the algebra isomorhic to $sl(2,R) \sim su(1,1) \sim so(1,2)$. These algebras are frequently used in quantum optics. The only non-zero coefficients of the Killing–Cartan tensor are $g_{12} = g_{21} = -2g_{33}$. Since $\det g \neq 0$, we have three invariants which depend on the *fourth-order moments* of the coordinate and momentum operators,

$$\mathcal{G}_0 = R_{11}R_{22}R_{33} + 2R_{12}R_{23}R_{31} - R_{11}R_{23}^2 - R_{22}R_{31}^2 - R_{33}R_{12}^2 \tag{5.14}$$

$$\mathcal{G}_1 = R_{11}R_{22} - R_{12}^2 + 4\left(R_{12}R_{33} - R_{13}R_{23}\right) \tag{5.15}$$

$$\mathcal{G}_2 = R_{12} - R_{33} \tag{5.16}$$

as well as the related invariants $\mathcal{G}'_m$ and $\mathcal{G}''_m$. In particular, $\mathcal{G}''_2$ is the generalization of the invariant Δ' (2.13) for homogeneous one-dimensional quadratic Hamiltonians:

$$\mathcal{G}''_2 = R_1R_2 - R_3^2 = \langle\hat{p}^2\rangle\langle\hat{x}^2\rangle - \tfrac{1}{4}\langle\hat{p}\hat{x} + \hat{x}\hat{p}\rangle^2. \tag{5.17}$$

The invariant $\mathcal{G}'_1$ reads

$$\mathcal{G}'_1 = \langle\hat{p}^4\rangle\langle\hat{x}^4\rangle + \tfrac{3}{4}\left(\langle\hat{p}^2\hat{x}^2 + \hat{x}^2\hat{p}^2\rangle\right)^2 - \tfrac{3}{2}\hbar^2\langle\hat{p}^2\hat{x}^2 + \hat{x}^2\hat{p}^2\rangle - \langle\hat{p}^3\hat{x} + \hat{x}\hat{p}^3\rangle\langle\hat{p}\hat{x}^3 + \hat{x}^3\hat{p}\rangle.$$

Other examples can be found in [22, 24, 61].

6. Discussion

There are several areas in classical physics, where the dynamics is governed by an effective Schrödinger equation with oscillator-like potential, so that the transformation relating 'initial' and 'final' sets of variables describing the state of the system possesses a symplectic structure, which is a necessary prerequisite for the existence of universal invariants. In particular, such a situation takes place for paraxial optical beams.

Consider, for instance, the Helmholtz equation $\nabla^2 E + k^2 n^2(\boldsymbol{r})E = 0$, which describes the propagation of harmonic wave fields (here k is the wavenumber and $n(\boldsymbol{r})$ is the refractive index of the medium). Making the substitution

$$E(\boldsymbol{r}) = n_0^{-1/2}\psi(x, y; z)\exp\left(\mathrm{i}k\int^z n_0(\xi)\,\mathrm{d}\xi\right) \tag{6.1}$$

and neglecting the second derivatives of ψ with respect to the longitudinal coordinate z, as well as the derivatives of the function $n_0(z)$, one arrives at the equation [62, 63]

$$\mathrm{i}\tilde{\lambda}\frac{\partial\psi}{\partial z} = -\frac{\tilde{\lambda}^2}{2n_0}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{1}{2n_0}\left[n_0^2(z) - n^2(x, y, z)\right]\psi \tag{6.2}$$

where $\tilde{\lambda} = \lambda/2\pi = k^{-1}$ is the reduced wavelength in vacuum and $n_0 \equiv n(0, 0, z)$ is the refractive index of the medium on the axis of the beam propagating in the z-direction. Obviously, the parabolic equation (6.2) can be considered as an effective Schrödinger equation, provided one identifies $\tilde{\lambda}$ with the 'effective Planck constant' and the longitudinal coordinate z with 'time'. The 'momentum' operator in this case reads as $\hat{\boldsymbol{p}} = -\mathrm{i}\tilde{\lambda}\partial/\partial\boldsymbol{x}$. Consequently, the universal invariants exist for paraxial beams propagating in the media with a quadratic dependence of the dielectric permittivity n^2 on the transverse coordinates: $n^2 - n_0^2 = a_{ik}(z)x_i x_k + b_i(z)x_i$, where the coefficients a_{ik} and b_i may be rather arbitrary real functions of the longitudinal coordinate z (provided $(\tilde{\lambda}/n_0^2)|\mathrm{d}n_0/\mathrm{d}z| \ll 1$, which is the necessary condition of validity of the parabolic approximation). For the first time the existence of such invariants and their general structure (including higher-order moments) for paraxial optical beams of an arbitrary shape was established in [23]. The simplest examples related to a particular case of Gaussian beams were considered in [64, 65]. For such beams, the conservation of the invariant Δ (2.13) means that the ratio of the correlation radius to the width of the beam remains constant. This result was obtained for the Gaussian beams propagating in the *free space* in [66], for discrete linear optical systems in [64], and in the most general case in [23]. In the optical literature, the invariant Δ is sometimes called the '*beam quality factor*' [30, 31] (more frequently, this name is used for the 'uncertainty product' $M^2 = \sigma_{pp}\sigma_{xx}$ taken at the waist cross section of the beam [67]). The influence of the non-parabolic refraction index profile on the conservation of the invariants was studied in [23]. The idea of using universal invariants for the classification of nonlinearities was formulated in [27, 68].

Making the substitution (6.1) in the wave equation $(n^2/c^2)\partial^2 E/\partial t^2 - \nabla^2 E = 0$ and neglecting the same terms as in the case of the Helmholtz equation, we obtain another parabolic equation [69]

$$\mathrm{i}\tilde{\lambda}\frac{\partial\psi}{\partial z} = \frac{\tilde{\lambda}^2}{2n}\left(\frac{n_0^2}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) + \frac{n}{2}\psi. \tag{6.3}$$

It can be considered as the Schrödinger equation with a quadratic Hamiltonian, if the refraction index $n = n_0$ does not depend on the transverse coordinates x, y and time t, although an arbitrary (not too fast) dependence $n_0(z)$ is permitted. Note that the effective Hamiltonian is not non-negatively definite in this case; nonetheless, this fact is unimportant for the existence of the universal invariants, which include moments like

$$\overline{t^2} = \int \psi^*(x, y, t) t^2 \psi(x, y, t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t$$

or

$$\overline{p_t^2} = -\tilde{\lambda}^2 \int \psi^*(x, y, t) \frac{\partial^2}{\partial t^2} \psi(x, y, t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t.$$

Here we assume that the beam has a finite extension not only in space, but in time, as well. Explicit expressions for the 'spacetime' optical invariants were given in [69] and recently in [70].

Parabolic equations such as (6.2) were applied to describe the transverse (classical) motion in accelerator beams [71–74]. In this case the 'effective Planck constant' is proportional to the *emittance*, i.e. the phase space volume of the beam. Another example is the Fock–Schwinger proper-time representation of the relativistic Klein–Gordon or Dirac equations for a quantum charged particle in external electromagnetic fields [18, 75]. In all of these cases one can construct universal invariants similar to those given in the preceding sections.

The universal invariants are intimately related to the generalizations of the uncertainty relations [19, 22, 49]. Moreover, they can be extended to the case of more general commutation relations than (2.1): there exist constructions based on the anticommutators (fermionic operators) [22], or even on some mixtures of the commutators and anticommutators (Fermi–Bose systems) [19]. It would be interesting to try to generalize these constructions to the case of parastatistics or to the case of q-deformed commutation relations. Besides, it would be interesting to find the new invariants considering the Schrödinger-like dynamic equations for the quantum quasiprobabilities different from the Wigner function, such as the Cahill–Glauber distributions [76], the 'ambiguity function' [77] or the 'extended Wigner function' [78].

An intriguing problem is related to the Schrödinger-like equations with *non-Hermitian* effective quadratic Hamiltonians. Such Hamiltonians arise naturally, for example, if one adds to the right-hand side of equation (3.6) terms with second derivatives, transforming it into the Fokker–Planck equation, which can be used for a phenomenological description of relaxation in quantum systems [79]. Another example is the parabolic equation for the mutual coherence function of a beam propagating through a stochastic (turbulent) medium [80]. There is no problem in constructing the generalizations of the 'dynamical' invariants (which depend on the concrete form of the Hamiltonian) to the non-Hermitian case [58]. However, the direct generalization of the *universal invariants* appeared impossible due to the non-conservation of the standard normalization $\int \psi^* \psi\, \mathrm{d}x$ [24]. Perhaps, 'non-standard' definitions of the average values and normalizations, like those discussed at the end of section 3, could help to solve this problem.

References

[1] Lewis H R Jr and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[2] Malkin I A, Man'ko V I and Trifonov D A 1969 *Phys. Lett.* A **30** 414
[3] Malkin I A, Man'ko V I and Trifonov D A 1970 *Phys. Rev.* D **2** 1371
[4] Dodonov V V, Malkin I A and Man'ko V I 1972 *Physica* **59** 241
[5] Dodonov V V and Man'ko V I 1979 *Phys. Rev.* A **20** 550
[6] Dodonov V V, Kurmyshev E V and Man'ko V I 1980 *Phys. Lett.* A **79** 150
[7] Dodonov V V, Malkin I A and Man'ko V I 1975 *J. Phys. A: Math. Gen.* **8** L19
[8] Dodonov V V, Malkin I A and Man'ko V I 1975 *Int. J. Theor. Phys.* **14** 37
[9] Dodonov V V, Malkin I A and Man'ko V I 1974 *Physica* **72** 597

[10] Courant E D and Snyder H D 1958 *Ann. Phys., NY* **3** 1
[11] Lewis H R Jr 1967 *Phys. Rev. Lett.* **18** 510
[12] Camiz P, Gerardi A, Marchioro C, Presutti E and Scacciatelli E 1971 *J. Math. Phys.* **12** 2040
[13] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[14] Morales D A 1988 *J. Phys. A: Math. Gen.* **21** L889
[15] Dodonov V V and Man'ko V I 1989 Topological phases in quantum theory *Proc. Int. Seminar (Dubna, 1988)* ed B Markovski and S I Vinitsky (Singapore: World Scientific) p 74
[16] Mizrahi S S 1989 *Phys. Lett.* A **138** 465
[17] Leach P G L 1990 *J. Phys. A: Math. Gen.* **23** 2695
[18] Dodonov V V, Malkin I A and Man'ko V I 1976 *Physica* A **82** 113
[19] Dodonov V V and Man'ko V I 1989 Invariants and the evolution of nonstationary quantum systems *Proc. Lebedev Physics Institute* vol 183 (Commack: Nova Science)
[20] Gantmacher F 1970 *Lectures in Analytical Mechanics* (Moscow: Mir)
[21] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)
[22] Dodonov V V and Man'ko V I 1983 Group theoretical methods in physics *Proc. 2nd Int. Seminar (Zvenigorod, 24–6 November 1982)* vol 2, ed M A Markov, V I Man'ko and A E Shabad (Moscow: Nauka) p 11 (in Russian)
Dodonov V V and Man'ko V I 1985 Group theoretical methods in physics *Proc. 2nd Int. Seminar (Zvenigorod, 24–6 November 1982)* vol 1, ed M A Markov, V I Man'ko and A E Shabad (Geneva: Harwood) p 591
[23] Dodonov V V and Man'ko O V 1986 Group theoretical methods in physics *Proc. 3rd Seminar (Yurmala, 1985)* vol 2, ed V V Dodonov, M A Markov and V I Man'ko (Utrecht: VNU Science) p 523
[24] Dodonov V V and Man'ko V I 1987 Group theory, gravitation and elementary particle physics *Proc. Lebedev Physics Institute* vol 167, ed A A Komar (Commack: Nova Science) p 7
[25] Holm D D, Lysenko W P and Scovel J C 1990 *J. Math. Phys.* **31** 1610
[26] Neri F and Rangarajan G 1990 *Phys. Rev. Lett.* **64** 1073
[27] Dragt A J, Neri F and Rangarajan G 1992 *Phys. Rev.* A **45** 2572
[28] Bastiaans M J 1991 *Optik* **88** 163
[29] Kauderer M 1991 *Appl. Opt.* **30** 1025
[30] Serna J, Martínez-Herrero R and Mejías P M 1991 *J. Opt. Soc. Am.* A **8** 1094
[31] Belanger P A 1991 *Opt. Lett.* **16** 196
[32] Onciul D 1993 *J. Opt. Soc. Am.* A **10** 295
[33] Lin Q, Wang S, Alda J and Bernabeu E 1993 *Opt. Lett.* **18** 669
[34] Nemes G and Siegman A E 1994 *J. Opt. Soc. Am.* A **11** 2257
[35] Dragoman D 1994 *J. Opt. Soc. Am.* A **11** 2643
[36] Martínez-Herrero R and Mejías P M 1997 *Opt. Commun.* **140** 57
[37] Yang J and Fan D 1999 *J. Opt. Soc. Am.* A **16** 2488
[38] Gantmakher F R 1966 *The Theory of Matrices* (Moscow: Nauka)
[39] Hernández E S and Remaud B 1980 *Phys. Lett.* A **75** 269
[40] Turner R E and Snider R F 1981 *Can. J. Phys.* **59** 457
[41] Dodonov V V, Man'ko V I and Rudenko V N 1980 *Sov. J. Quantum Electron* **10** 1232
[42] Tatarskii V I 1983 *Sov. Phys.–Usp.* **26** 311
[43] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 121
[44] Littlejohn R G 1986 *Phys. Rep.* **138** 193
[45] Dodonov V V, Man'ko O V and Man'ko V I 1994 *Phys. Rev.* A **50** 813
[46] Dodonov V V, Man'ko O V and Man'ko V I 1994 *Phys. Rev.* A **49** 2993
[47] Snygg J 1980 *Am. J. Phys.* **48** 964
[48] Takabayasi T 1954 *Prog. Theor. Phys.* **11** 341
[49] Robertson H P 1934 *Phys. Rev.* **46** 794
[50] Dodonov V V, Man'ko O V and Man'ko V I 1995 *J. Russian Laser Res.* **16** 1
[51] Dattoli G, Mari C, Richetta M and Torre A 1992 *Nuovo Cimento* B **107** 269
[52] Schrödinger E 1930 *Ber. Kgl. Akad. Wiss. Berlin* **24** 296
[53] Robertson H P 1930 *Phys. Rev.* **35** 667
[54] Remaud B and Hernandez E S 1980 *Physica* A **103** 35
[55] Groenewold H J 1946 *Physica* **12** 405
[56] Akhundova E A, Dodonov V V and Man'ko V I 1985 *Theor. Math. Phys.* **60** 907
[57] Dodonov V V and Man'ko V I 1986 *Physica* A **137** 306
[58] Dodonov V V and Man'ko V I 1978 *Physica* A **94** 403
[59] Chountasis S and Vourdas A 1998 *Phys. Rev.* A **58** 1794

[60] Barut A O and Rączka R 1986 *Theory of Group Representations and Applications* (Singapore: World Scientific)
[61] Dodonov V V, Man'ko V I and Zhivotchenko D V 1993 *Nuovo Cimento* B **108** 1349
[62] Leontovich M A 1944 *Izv. Akad. Nauk SSSR* **8** 16
[63] Arnaud J A 1976 *Beam and Fiber Optics* (New York: Academic)
[64] Simon R, Sudarshan E C G and Mukunda N 1984 *Phys. Rev.* A **29** 3273
[65] Simon R, Sudarshan E C G and Mukunda N 1985 *Phys. Rev.* A **31** 2419
[66] Collett E and Wolf E 1980 *Opt. Commun.* **32** 27
[67] Siegman A E 1991 *IEEE J. Quantum Electron* **27** 1146
[68] Atakishiyev N M, Chumakov S M, Rivera A L and Wolf K B 1996 *Phys. Lett.* A **215** 128
[69] Dodonov V V and Man'ko O V 1987 *Computing Optics. Fundamentals* ed E P Velikhov and A M Prokhorov (Moscow: International Center of Scientifical and Technical Information) p 84
[70] Cao Q and Deng X M 1997 *Opt. Commun.* **142** 135
[71] Fedele R and Miele G 1991 *Nuovo Cimento* D **13** 1527
[72] Dattoli G, Giannessi L, Mari C, Richetta M and Torre A 1992 *Opt. Commun.* **87** 175
[73] Fedele R and Man'ko V I 1998 *Phys. Rev.* E **58** 992
[74] Fedele R, Man'ko M A and Man'ko V I 2000 *J. Russian Laser Res.* **21** 1
[75] Bagrov V G, Buchbinder I L and Gitman D M 1976 *J. Phys. A: Math. Gen.* **9** 1955
[76] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
[77] Papoulis A 1974 *J. Opt. Soc. Am.* **64** 779
[78] Chountasis S and Vourdas A 1999 *J. Phys. A: Math. Gen.* **32** 6949
[79] Dodonov V V and Man'ko O V 1985 *Physica* A **130** 353
[80] Rytov S M, Kravtsov Yu A and Tatarskii V I 1978 *Introduction to Statistical Radiophysics* (Moscow: Nauka)